Lecture 32.

Recall from last time that for variables X, Y,

$$\operatorname{cov}(X,Y) := E[(X-E[X])(Y-E[Y])]$$

and that $\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y]$.

<u>Prop:</u> $\operatorname{cov}(X,Y) = \operatorname{cov}(Y,X)$.

<u>Pf:</u> obvious since $XY = YX$, so
$$E[XY] - E[X]E[Y] = E[YX] - E[Y]E[X]$$
$\square$

<u>Prop:</u> $\operatorname{Cov}(X,X) = \operatorname{Var}(X)$.

<u>Pf:</u> $\operatorname{Cov}(X,X) = E[(X-E(X))(X-E[X])] = E[(X-E[X])^2]$
$= \operatorname{Var}(X)$. $\square$

<u>Prop:</u> $\operatorname{Cov}(aX,Y) = a\operatorname{Cov}(X,Y)$

<u>Pf:</u>
$$\begin{aligned}\operatorname{Cov}(aX,Y) &= E[aXY] - E[aX]E[Y] \\ &= aE[XY] - aE[X]E[Y] \\ &= a(E[XY] - E[X]E[Y]) = a\operatorname{Cov}(X,Y).\end{aligned}$$
$\square$

Prop: $\operatorname{Cov}\left(\sum_{i=1}^{n} X_i,\ \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{Cov}(X_i, Y_j)$

Pf: Observe that it suffices to prove that

$$\operatorname{Cov}\left(\sum_{i=1}^{n} X_i, Y\right) = \sum_{i=1}^{n} \operatorname{Cov}(X_i, Y). \quad *$$

Then substituting $Y = Y_1 + \cdots + Y_n$ and the fact that $\operatorname{Cov}(X,Y) = \operatorname{Cov}(Y,X)$ gives the desired result.

Let $\mu_i = E[X_i]$, $E[Y] = \nu$

Then
$$\begin{aligned}
\operatorname{Cov}\left(\sum_{i=1}^{n} X_i, Y\right) &= E\left[\left(\sum_{i=1}^{n} X_i\right) Y\right] - E\left[\sum_i X_i\right] E[Y] \\
&= E\left[\sum_i (Y X_i)\right] - E\left[\sum_i X_i\right] E[Y] \\
&= \sum_{i=1}^{n} E[X_i Y] - \sum_{i=1}^{n} E[X_i] E[Y] \\
&= \sum_{i=1}^{n} \left(E[X_i Y] - E[X_i] E[Y]\right) \\
&= \sum_{i=1}^{n} \operatorname{Cov}(X_i, Y) \text{ as required.}
\end{aligned}$$

Corollary (Variance of a sum of RV's).

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2 \sum_{i<j} \operatorname{Cov}(X_i, X_j).$$

Proof: $\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \operatorname{Cov}\left(\sum_{i=1}^{n} X_i,\ \sum_{i=1}^{n} X_i\right)$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \Big(\underbrace{\mathrm{Cov}(X_i, X_i)}_{= \mathrm{Var}(X_i)} + \sum_{\{j : j \neq i\}} \mathrm{Cov}(X_i, X_j) \Big)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j). \qquad \square$$

<u>Corollary</u>: Let $\{X_1, \dots, X_n\}$ be independent.
Then $\mathrm{Var}(X_1 + \dots + X_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$.

<u>Pf</u>: By independence, $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$.
The result follows.

<u>Correlation</u>:

For random variables X and Y, the correlation of X and Y is defined to be

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

Proposition: $\forall X, Y, \quad -1 \leq \rho(X,Y) \leq 1$.

Pf: Let $Var(X) = \sigma_X^2 \quad Var(Y) = \sigma_Y^2$.

Then $\forall n$,
$$0 \leq Var\left(\frac{X}{\sigma_X} + (-1)^n \frac{Y}{\sigma_Y}\right)$$
$$= \frac{Var(X)}{\sigma_X^2} + \frac{Var(Y)}{((-1)^n \sigma_Y)^2} + (-1)^n 2 \underbrace{\frac{Cov(X,Y)}{\sigma_X \sigma_Y}}_{\rho(X,Y)}$$
$$= 1 + 1 + (-1)^n 2 \rho(X,Y)$$
$$= 2(1 + (-1)^n \rho(X,Y)).$$

So forall n,
$$0 \leq 1 + (-1)^n \rho(X,Y).$$
$$-1 \leq (-1)^n \rho(X,Y).$$

For n even, we get $-1 \leq \rho(X,Y)$.
For n odd, we get $-1 \leq -\rho(X,Y)$, so $1 \geq \rho(X,Y)$.

Hence $\quad -1 \leq \rho(X,Y) \leq 1$.

$\square$

Proposition: $\rho(X,Y) = \pm 1$ iff $Y = a \pm |b| X$.

Pf: Left as an exercise.

– The idea is that $\rho(X,Y)$ "measures" how linearly related ~~X~~ and Y are.

Moment Generating Functions:

Let X be a random variable.

Then we define the moment generating function of X as

$$m_X(t) = E[e^{tX}] \quad \text{for } t \in \mathbb{R}.$$

So

$$m_X(t) = \sum_x e^{tx} p(x) \qquad \text{if } X \text{ is discrete}$$

and

$$m_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \qquad \text{if } X \text{ is continuous.}$$

Let's study the continuous case. By Leibniz's rule

$$\frac{d^n}{dt^n} m_X(t) = m_X^{(n)}(t) = \int_{-\infty}^{\infty} \frac{d^n}{dt^n} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^n e^{tx} f(x)\,dx.$$

and so $\quad m_X^{(n)}(0) = \int_{-\infty}^{\infty} x^n f(x)\,dx = E[X^n]$, the n^{th} moment.

The proof is identical in the discrete case.

<u>Ex:</u> Let $X = \mathrm{Bin}(n, p)$.

Then

$$\begin{aligned} m_X(t) &= E\left[e^{tX}\right] \\ &= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \left(e^t\right)^k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(pe^t\right)^k (1-p)^{n-k} \\ &= \left(pe^t + 1 - p\right)^n \quad \left(\text{by the Binomial theorem}\right) \end{aligned}$$